

A GENERALIZATION OF TVERBERG'S THEOREM

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ABSTRACT. The well known theorem of Tverberg states that if $n \geq (d+1)(r-1)+1$ then one can partition any set of n points in \mathbb{R}^d to r disjoint subsets whose convex hulls have a common point. The numbers $T(d, r) = (d+1)(r-1)+1$ are known as Tverberg numbers. Reay asks the following question: if we add an additional parameter k ($2 \leq k \leq r$) what is the minimal number of points we need in order to guarantee that there exists an r partition of them such that any k of the r convex hulls intersect. This minimal number is denoted by $T(d, r, k)$. Reay conjectured that $T(d, r, k) = T(d, r)$ for all d, r and k . In this article we prove that this is true for the following cases: when $k \geq \lfloor \frac{d+3}{2} \rfloor$ or when $d < \frac{rk}{r-k} - 1$ and for the specific values $d = 3, r = 4, k = 2$ and $d = 5, r = 3, k = 2$.

1. INTRODUCTION

A well known theorem of Radon says that any set of $d+2$ or more points in \mathbb{R}^d can be partitioned into two disjoint parts whose convex hulls meet. This follows easily from the fact that every set of $d+2$ points in \mathbb{R}^d is affinely dependent.

The corresponding statement for partitions into more than two parts is known as Tverberg's theorem, and is much more difficult.

Theorem 1.0.1. (*H. Tverberg, 1966*) *Let a_1, \dots, a_n be points in \mathbb{R}^d . If $n > (d+1)(r-1)$ then the set $N = \{1, \dots, n\}$ of indices can be partitioned into r disjoint parts N_1, \dots, N_r in such a way that the r convex hulls $\text{conv}\{a_i : i \in N_j\}$ ($j=1, \dots, r$) have a point in common.*

(This formulation covers also the case where the points a_1, \dots, a_n are not all distinct.) Henceforth we use the abbreviation $a(N_j)$ for $\{a_i : i \in N_j\}$, and $[a(N_j)]$ for $\text{conv}(a(N_j))$. The original proof (see [Tv66]) was quite difficult. In 1981, Tverberg published another proof of theorem 1.0.1 much simpler than the original proof (see [Tv81]). One of the simplest proofs of theorem 1.0.1 known today is due to Sarkaria ([SK]).

The numbers $T(d, r) = (d+1)(r-1)+1$ are known as Tverberg numbers. The condition $n \geq T(d, r)$ in Tverberg's theorem is extremely tight. If $n < T(d, r)$, then almost always, for any r -partition N_1, \dots, N_r of the set $N = \{1, \dots, n\}$, even the intersection of the **affine** hulls $\text{aff}(a(N_j))$ ($j = 1, \dots, r$) is empty.

In fact, there exists a polynomial P , not identically zero, in $n \cdot d$ scalar variables $P(\vec{x}_1, \dots, \vec{x}_n) = P(x_{11}, \dots, x_{1d}, \dots, x_{n1}, \dots, x_{nd})$ such that, for any r -partition N_1, \dots, N_r of N , $\cap_{j=1}^r \text{aff}(a(N_j)) = \emptyset$, unless $P(a_1, \dots, a_n) = 0$.

In this paper we weaken the condition $\cap_{j=1}^r [a(N_j)] \neq \emptyset$ ($j = 1, \dots, r$) in Tverberg's theorem and ask only that each k of the convex hulls $[a(N_j)]$ ($j = 1, \dots, r$) meet, where k is an additional parameter, $2 \leq k \leq r$, this weakened condition may perhaps require fewer than $T(d, r)$ points. Thus we define $T(d, r, k)$ to be the

smallest positive integer n with the following property: for any list a_1, \dots, a_n of points in \mathbb{R}^d , there is an r -partition N_1, \dots, N_r of the set of indices $N = \{1, \dots, n\}$ such that every k of the r convex hulls $[a(N_j)]$ have a point in common.

The function $T(d, r, k)$ is clearly monotone non-decreasing in each of the parameters d, r, k and $T(d, r, r) = T(d, r)$.

If $r > d + 1$, and each $d + 1$ of the convex hulls $[a(N_j)]$ ($j = 1, \dots, r$) meet, then they all meet, by Helly's theorem. Thus $T(d, r, k) = T(d, r)$ for $d + 1 \leq k \leq r$. This reduces the interesting range of k to $2 \leq k \leq \min(r - 1, d)$.

John Reay (see [Ry]) settled the case $d = 2$, showing that $T(2, r, 2) = T(2, r)$ for all $r \geq 2$. He also showed that $T(3, 3, 2) = T(3, 3) (= 9)$ and made the following bold conjecture.

Conjecture 1.0.2. $T(d, r, k) = T(d, r)$ for all $2 \leq k \leq r$.

We don't really believe this is true. To press our point, consider the case $d = r = 1000$. By Tverberg's theorem, a million points in \mathbb{R}^{1000} can be partitioned into one thousand parts whose convex hulls have a common point. Is there a set of 999,999 points in \mathbb{R}^{1000} that cannot be partitioned into 1000 parts whose convex hulls intersect just pairwise? Seems implausible.

Nevertheless, the purpose of this paper is to establish parts of Reay's conjecture. We show, by means of suitable examples, that Reay's conjecture does hold for the following cases:

Theorem 1.0.3. For every dimension $d \geq 3$ and for every $r \geq \lfloor \frac{d+3}{2} \rfloor$:

$$T(d, r, \lfloor \frac{d+3}{2} \rfloor) = T(d, r) = (d+1)(r-1) + 1$$

In particular the above shows that $T(3, 4, 3) = T(3, 4) = 13$. For $d = 3, r = 4, k = 2$ we have the following theorem:

Theorem 1.0.4. $T(3, 4, 2) = T(3, 4) = 13$.

Another general family of cases is described in:

Theorem 1.0.5. For every $k < r$ and for every dimension $d < \frac{kr}{r-k} - 1$:

$$T(d, r, k) = T(d, r) = (d+1)(r-1) + 1$$

Therefore, if $r = 3, k = 2$ then for $d < 5$ it is clear that $T(d, r, k) = T(d, r)$. In the case where $d = 5$ we have the following:

Theorem 1.0.6. $T(5, 3, 2) = T(5, 3) = 13$.

2. PROOF OF THEOREM 1.0.3

For the proof we will use the following (counter) example: let $p_0, p_1, \dots, p_d \in \mathbb{R}^d$ be the vertices of a d -simplex centered at the origin, i.e., $\sum_{i=0}^d p_i = 0$ and each d of the points p_0, p_1, \dots, p_d are linearly independent. Denote $D = \{0, 1, \dots, d\}$ and for $i \in D$ define $R_i = \{\lambda p_i : \lambda > 0\}$ (the open ray emanating from $\mathbf{0}$ through p_i).

On each ray R_i we choose $r - 1$ distinct points. The chosen points form a set $X \subset \mathbb{R}^d$, $|X| = (d+1)(r-1) = T(d, r) - 1$. We are going to show that in every r -partition of X into r subsets ($X = C_1 \cup \dots \cup C_r$) we can find k among them, $k \leq \lfloor \frac{d+3}{2} \rfloor$, whose convex hulls have empty intersection. This will show that $T(d, r, k) = T(d, r)$ for $\lfloor \frac{d+3}{2} \rfloor \leq k \leq r$. We start with some preliminaries concerning the "positive basis" $P = \{p_0, p_1, \dots, p_d\}$ of \mathbb{R}^d .

2.1. Properties of the spanning set $P = \{p_0, p_1, \dots, p_d\}$.

Proposition 2.1.1. *Every point $x \in \mathbb{R}^d$ has a representation*

$$(2.1.1) \quad x = \xi_0 p_0 + \xi_1 p_1 + \dots + \xi_d p_d$$

where $\min\{\xi_0, \xi_1, \dots, \xi_d\} = 0$. This representation is unique. We call it the non-negative representation of the point.

Proof. The vectors p_0, p_1, \dots, p_d span \mathbb{R}^d linearly. In fact, each d of them form a linear basis of \mathbb{R}^d . Let $x = \sum_{i=0}^d \alpha_i p_i$ be some fixed representation of x in terms of P . The most general linear dependence among p_0, p_1, \dots, p_d is $\sum_{i=0}^d \lambda p_i = 0$, $\lambda \in R$. Therefore the most general representation of x in terms of P is $x = \sum_{i=0}^d (\alpha_i - \lambda) p_i$, $\lambda \in R$. To obtain a representation with the smallest coefficient equal 0, we must choose $\lambda = \min\{\alpha_i : i \in D\}$. \square

We call 2.1.1 the non-negative representation of x (in terms of P). The support of x , (with respect to P) is defined by

$$\text{supp } x = \{i \in D : \xi_i > 0\}.$$

Simple properties of $\text{supp } x$:

- (1) $\emptyset \subseteq \text{supp } x \subset D$.
- (2) $\text{supp } x = \emptyset$ iff $x = \underline{0}$.
- (3) $\text{supp } p_i = \{i\}$.
- (4) $\text{supp } \lambda x = \text{supp } x$ for $\lambda > 0$.
- (5) $\text{supp } (x + y) \subseteq \text{supp } x \cup \text{supp } y$, with equality iff $\text{supp } x \cup \text{supp } y \neq D$.
- (6) If $x \neq 0$, then $\text{supp } x \cup \text{supp } (-x) = D$.

Recall that our set X consists of $r - 1$ distinct points on each ray R_i ($i \in D$). For a subset $C \subset X$, define $I(C) = \{i \in D : C \cap R_i \neq \emptyset\}$. Now make the following observations:

Proposition 2.1.2. *If $C \subset X$ and $x \in \text{conv}(C)$, then $\text{supp } x \subset I(C)$. (This is obviously true also when $I(C) = D$.)*

When I is a subset of D , we shall denote by $R(I)$ the union $\cup\{R_i : i \in I\}$.

Proposition 2.1.3. *Suppose $C \subset X$, and $x \in \text{conv}(C)$. If $I(C) \neq D$ then $x \in \text{conv}\{C \cap R(\text{supp } x)\}$.*

Proof. Suppose $x = \sum_{\nu=1}^n \gamma_\nu c_\nu$, where $c_\nu \in C$, $\gamma_\nu > 0$, $\sum_{\nu=1}^n \gamma_\nu = 1$. If $c_\nu = \lambda_\nu p_i \in R_i$, $\lambda_\nu > 0$, then p_i will appear with a positive coefficient in the non-negative representation of x in terms of P , and therefore $i \in \text{supp } x$. Note that we have used the fact that $I(C) \neq D$. \square

For points $a = \alpha p_i \in R_i$, $b = \beta p_i \in R_i$, ($\alpha, \beta > 0$) we say that a is lower than b on R_i if $\alpha < \beta$ (or, equivalently, if $\|a\| < \|b\|$).

Proposition 2.1.4. *Suppose $I \subsetneq D$. Let C, C' be two finite subsets of $R(I) (= \cup\{R_i : i \in I\})$. If, for each $i \in I$, every point of $C \cap R_i$ is lower (on R_i) than every point of $C' \cap R_i$, then $\text{conv}(C) \cap \text{conv}(C') = \emptyset$.*

Proof. Assume, w.l.o.g., that $|I| = d$. (We do not assume that $C \cap R_i \neq \emptyset$ and $C' \cap R_i \neq \emptyset$ for all $i \in I$.) For each $i \in I$ choose a point $s_i = \sigma_i p_i \in R_i$ that is higher (on R_i) than every point of $C \cap R_i$ and lower than every point of $C' \cap R_i$. The d points

s_i ($i \in I$) are linearly independent, and their affine hull $H = \text{aff}\{s_i : i \in I\} \subset \mathbb{R}^d$ is a hyperplane that does not pass through the origin. Denote by H_-, H_+ the two open half spaces determined by H , and assume $0 \in H_-$. From our assumptions it follows that $C \subset H_-$ and $C' \subset H_+$, hence $\text{conv}(C) \cap \text{conv}(C') = \emptyset$. \square

Proposition 2.1.5. *Suppose $U \subsetneq D$. Let $C_1, C_2, C_3, C_4, \dots, C_n$ be subsets of X . Assume*

- (1) $I(C_\nu) \subsetneq D$ for $\nu = 1, 2, \dots, n$.
- (2) $\cap_{\nu=1}^n I(C_\nu) \subset U$.
- (3) For each $i \in U$, each point of $C_1 \cap R_i$ is lower (on R_i) than every point of $C_2 \cap R_i$.

Then $\cap_{\nu=1}^n \text{conv}(C_\nu) = \emptyset$.

Proof. Assume, on the contrary, that $\cap_{\nu=1}^n \text{conv}(C_\nu) \neq \emptyset$, and assume $x \in \cap_{\nu=1}^n \text{conv}(C_\nu)$. By Proposition 2.1.2 we conclude that $\text{supp } x \subset \cap_{\nu=1}^n I(C_\nu) \subset U$. Applying proposition 2.1.3 to C_1 and C_2 , we find that

$$(2.1.2) \quad x \in \text{conv}(C_\nu \cap R(U)) \text{ for } \nu = 1, 2.$$

Now invoke proposition 2.1.4 with $C = C_1 \cap R(U)$, $C' = C_2 \cap R(U)$, and $I = U$, to conclude that $\text{conv}(C_1 \cap R(U)) \cap \text{conv}(C_2 \cap R(U)) = \emptyset$, which contradicts 2.1.2. \square

2.2. Completion of the proof. Let $X \subset \mathbb{R}^d$ be the set described in section 1 ($r-1$ points on each of the rays R_0, R_1, \dots, R_d), and let C_1, \dots, C_r be an arbitrary partition of X into r disjoint sets. Our aim is to apply proposition 2.1.5 to some n of the parts C_i , with n as small as possible. We shall be able to do this with $n = \lceil \frac{d+3}{2} \rceil$.

Suppose $C_1, C_2 \subset X$, $I(C_1) \subsetneq D$, $I(C_2) \subsetneq D$, and $U \subset I(C_1) \cap I(C_2)$ satisfies that for each $i \in U$, each point of $C_1 \cap R_i$ is lower (on R_i) than every point of $C_2 \cap R_i$. For each $i \in I(C_1) \cap I(C_2) \setminus U$ denote by $C(i)$ a set that disjoint to R_i (Such set does exist since $|X \cap R_i| = r-1$). Notice that the sets $C(i)_{i \in I(C_1) \cap I(C_2) \setminus U}$ are not necessarily different.

In these conditions, the sets $C_1, C_2, \{C(i) : i \in I(C_1) \cap I(C_2) \setminus U\}$ satisfy the assumptions of proposition 2.1.5 and therefore $[C_1] \cap [C_2] \cap \cap_{i \in I(C_1) \cap I(C_2) \setminus U} [C(i)] = \emptyset$.

We will show that for each r -partition of X , there are two sets, w.l.o.g C_1, C_2 such that $|I(C_1) \cap I(C_2) \setminus U| \leq \lceil \frac{d-1}{2} \rceil$ (U is defined as aforementioned and depends on C_1, C_2). Therefore the sets $C_1, C_2, \{C(i) : i \in I(C_1) \cap I(C_2) \setminus U\}$ are $2 + |I(C_1) \cap I(C_2) \setminus U| \leq \lceil \frac{d+3}{2} \rceil$ sets whose convex hulls don't meet.

Proposition 2.2.1. *For every nonempty subset J of D there is a part C_i such that $|C_i \cap R(J)| \leq \lceil \frac{r-1}{r} |J| \rceil$*

Proof. $(r-1)|J| = |X \cup R(J)| = |(\cup_{i=1}^r C_i) \cap R(J)| = |(\cup_{i=1}^r C_i \cap R(J))| = \sum_{i=1}^r |C_i \cap R(J)|$.

Thus $|C_i \cap R(J)| \leq \frac{r-1}{r} |J|$ for some $0 \leq i \leq r$. \square

Assume C_1 is the smallest part. Then

$$|I(C_1)| \leq |C_1| \leq \lceil \frac{1}{r} |X| \rceil = \lceil \frac{r-1}{r} (d+1) \rceil \leq d$$

Now apply proposition 2.2.1 to the set $J = I(C_1)$ to find another part C_2 with

$$(2.2.1) \quad |C_2 \cap R(I(C_1))| \leq \frac{r-1}{r} |I(C_1)| < |I(C_1)|.$$

($C_2 \neq C_1$ since $|C_1 \cap R(I(C_1))| \geq |I(C_1)|$). Notice that particularly $|I(C_2)| \leq d$.

For $i = 1, 2$ we divide $I(C_i)$ to two disjoint sets:

$$S_i = \{j \in D : |C_i \cap R_j| = 1\}$$

$$M_i = \{j \in D : |C_i \cap R_j| > 1\}$$

and get:

$$(2.2.2) \quad |C_i| \geq |I(C_i)| + |M_i|.$$

Furthermore, for every subset J of D :

$$(2.2.3) \quad |C_i \cap R(J)| \geq |I(C_i) \cap J| + |M_i \cap J|.$$

Assume $I(C_1) = d - a$ ($a \geq 0$). From 2.2.2 we get

$$|M_1| \leq |C_1| - |I(C_1)| \leq d - (d - a) = a$$

and from 2.2.1 we get

$$|C_2 \cap R(I(C_1))| \leq d - a - 1.$$

The set $S_1 \cap S_2$ can be divided to two disjoint sets:

$$U_1 = \{j \in S_1 \cap S_2 : C_1 \text{ is lower than } C_2 \text{ on } R_j\}$$

$$U_2 = \{j \in S_1 \cap S_2 : C_2 \text{ is lower than } C_1 \text{ on } R_j\}.$$

If $|U_1| \geq |U_2|$ we define $U = U_1$, otherwise we define $U = U_2$. Anyway we get $|U| \geq \frac{1}{2}|S_1 \cap S_2|$.

Proposition 2.2.2. *Under these notations*

$$|I(C_1) \cap I(C_2) \setminus U| \leq \left\lfloor \frac{d-1}{2} \right\rfloor$$

Proof. It is enough to prove that $2|I(C_1) \cap I(C_2) \setminus U| \leq d - 1$.

Indeed:

$$\begin{aligned} 2|I(C_1) \cap I(C_2) \setminus U| &= 2|I(C_1) \cap I(C_2)| - 2|U| \\ &\leq 2|I(C_1) \cap I(C_2)| - |S_1 \cap S_2| \\ &= 2|I(C_1) \cap I(C_2)| - |(I(C_1) \setminus M_1) \cap S_2| \\ &= 2|I(C_1) \cap I(C_2)| - |I(C_1) \cap S_2| + |M_1 \cap S_2| \\ &\leq 2|I(C_1) \cap I(C_2)| - |I(C_1) \cap S_2| + |M_1| \\ &= |I(C_1) \cap I(C_2)| + (|I(C_1) \cap I(C_2)| - |I(C_1) \cap S_2|) + |M_1| \\ &= |I(C_1) \cap I(C_2)| + |I(C_1) \cap M_2| + |M_1| \\ &\leq_{\text{by (2.2.3)}} |C_2 \cap R(I(C_1))| + |M_1| \\ &\leq d - a - 1 + a \\ &= d - 1 \end{aligned}$$

□

In summary, we have proved that for every $d \geq 3, r \geq \lceil \frac{d+3}{2} \rceil$ there is a set of $(d+1)(r-1)$ points in \mathbb{R}^d , which in any r -partition of it, there are $\lceil \frac{d+3}{2} \rceil$ parts whose convex hulls don't meet. This completes the proof of theorem 1.0.3.

3. PROOF OF THEOREM 1.0.4: THE CASE $d = 3, r = 4, k = 2$

Looking again at the construction of X described in the previous section for $d = 3$ and $r = 4$, we conclude by theorem 1.0.3 that $T(3, 4, 3) = T(3, 4) = 13$, but this construction doesn't prove that $T(3, 4, 2) = 13$. In fact, with the 12 points of X , we **can** build 4 rectangles which intersect pairwise.

In this section we will show, that by moving 2 of the 12 points, we get 12 points, that in any 4-partition of them, there are 2 sets whose convex hulls don't meet. This will prove that $T(3, 4, 2) = 13$. This case ($d = 3, r = 4, k = 2$) is the smallest case for which the number $T(d, r, k)$ was not known.

We have a set X of 12 points like in the previous section, i.e. 3 points on each ray R_0, R_1, R_2, R_3 . Looking at all the 4-partitions of X ($X = C_0 \cup C_1 \cup C_2 \cup C_3$), we say that a given partition is a "good partition" if the convex hulls of 2 of the subsets don't meet. We say that a given partition is a "bad partition" if $[C_i] \cap [C_j] \neq \emptyset$ for all $0 \leq i < j \leq 3$.

Remark 3.0.3. *If we perturb the points of X a little bit (i.e. replace each point $x \in X$ by a point x' when $\|x - x'\| < \delta$ and δ is small enough), the good partitions remain good.*

We will see, that by moving 2 points, we can convert all the "bad partitions" to "good partitions".

Proposition 3.0.4. *Any 4-partition of X , such that $|I(C_0)| < 3$ is "good".*

Proof. There are two cases to consider: $|I(C_0)| = 1$ and $|I(C_0)| = 2$. If $|I(C_0)| = 1$ then there is a set, say C_1 , such that $I(C_1) \cap I(C_0) = \emptyset$ (This follows immediately from the fact that $|X \cap R(I(C_0))| = 3$). From proposition 2.1.2 we get that $[C_0] \cap [C_1] = \emptyset$.

If $|I(C_0)| = 2$ there are three cases:

- (1) $|C_0| \geq 4$. In this case $|(R(I(C_0)) \cap X) \setminus C_0| \leq 2$ and therefore there is a set C_i s.t. $I(C_i) \cap I(C_0) = \emptyset$. Again, from proposition 2.1.2 it follows that $[C_0] \cap [C_i] = \emptyset$.
- (2) $|C_0| = 3$. Here $|(R(I(C_0)) \cap X) \setminus C_0| = 3$ and each of the other sets contains one of the three points (otherwise, the same argument as in the previous case works). Therefore, there is a set C_i satisfying $I(C_i) \cap I(C_0) = \{j\}$ and $|C_0 \cap R_j| = |C_1 \cap R_j| = 1$. From proposition 2.1.5 with $U = \{j\}$ and $n = 2$ we get that $[C_0] \cap [C_i] = \emptyset$.
- (3) $|C_0| = 2$. This implies that $|(R(I(C_0)) \cap X) \setminus C_0| = 4$, so there is a set that contains only one of these points so we may use the argument of the previous case.

□

Henceforth we assume that $|I(C_i)| \geq 3$ for $0 \leq i \leq 3$ and therefore $|C_i| \geq 3$. Recalling that $|X| = 12$ we conclude that $|C_i| = 3$ for $0 \leq i \leq 3$. In other words, each part contains exactly 3 points from 3 different rays. It follows easily that associating each set C_i with the ray disjoint to C_i is a bijection. For convenience denote the ray disjoint to C_i by R_i .

Now, let's look at two sets: C_i, C_j where $I(C(i)) = \{j, k, l\}$ and $I(C(j)) = \{i, k, l\}$.

Proposition 3.0.5. *If in R_k, R_l , C_i is lower than C_j , then $[C_i] \cap [C_j] = \emptyset$.*

Proof. It follows from proposition 2.1.5 with $U = \{k, l\}$. \square

Definition 3.0.6. Denote the highest, middle and lowest points of X on the ray R_i by $p_{i,1}, p_{i,2}, p_{i,3}$ respectively.

Proposition 3.0.7. Each set C_i contains exactly one lowest point, one middle point, and one highest point.

Proof. If the lowest points of both R_k and R_l are in C_i , then due to proposition 3.0.5 this is not a "bad partition". For the same reason, C_i doesn't have two highest points. Therefore, the four lowest point are divided between the four sets, one for each, and the same is true for the highest points. Hence the middle points also obey this rule. \square

Now we are going to perturb the points of X in order to avoid intersection in the "bad" cases.

Step 1: Replace the point $p_{0,1}$ by $p'_{0,1} = p_{0,1} + \delta p_{3,1}$, where δ is small enough so the good partitions remain good. Now, let's look at $p'_{0,1}, p_{0,2}, p_{0,3}$. These points belong to C_1, C_2, C_3 since $C_0 \cap R_0 = \emptyset$. We claim that if $p'_{0,1} \in C_1$ or $p'_{0,1} \in C_2$, there are two sets whose convex hulls don't meet. Suppose $p'_{0,1} \in C_1$. In this case we can show that $[C_1] \cap [C_3] = \emptyset$. If there is $x \in [C_1] \cap [C_3]$ then from the fact that $x \in [C_1]$ we get that $\text{supp } x \subset \{0, 2, 3\}$ and since $x \in [C_3]$ we get that $\text{supp } x \subset \{0, 1, 2\}$. Therefore $\text{supp } x \subset \{0, 2\}$. But in $[C_1]$ there is only one point satisfying this condition. This point is one of the $p_{2,i}$ -s and it does not belong to $[C_3]$ which is absurd. In the case $p'_{0,1} \in C_2$, similar arguments show that $[C_2] \cap [C_3] = \emptyset$.

Let's suppose now that $p'_{0,1} \in C_3$. In this case what can we say about $p_{3,3}$? If $p_{0,2} \in C_1$, then C_1 is higher than C_2 on R_0 , therefore the highest point of R_3 cannot be in C_1 . From proposition 3.0.7 the middle point of R_3 cannot be in C_1 (since the middle point of R_0 is already in C_1). Therefore $p_{3,3} \in C_1$. Similarly, if $p_{0,2} \in C_2$ then $p_{3,3} \in C_2$.

Conclusion: $p_{3,3}$ is either in C_1 or in C_2 .

Step 2: Replace $p_{3,3}$ by $p'_{3,3} = p_{3,3} + \delta p_{0,3}$. (Again, δ is small enough not to destroy the "good partitions"). Repeating the arguments above, we conclude that if $p'_{3,3} \in C_1$ then $[C_1] \cap [C_3] = \emptyset$ and if $p'_{3,3} \in C_2$ then $[C_2] \cap [C_3] = \emptyset$.

After steps 1 and 2, we have a set of 12 points in \mathbb{R}^3 such that for any partition of it into 4 disjoint parts, there are two parts whose convex hulls do not meet. This completes the proof of theorem 1.0.4.

4. PROOF OF THEOREM 1.0.5

For this proof we use the same counter example as in theorem 1.0.3 with an additional restriction. Recall that we constructed a simplex with $d + 1$ vertices p_0, \dots, p_d whose center is at the origin. For each vertex p_i we defined R_i to be the open ray emanating from $\underline{0}$ through p_i . On each ray we chose $r - 1$ points. The additional restriction in this case is that the $r - 1$ points of each ray R_i ($i = 0, \dots, d$) are in general position. The union of all these points is denoted by $X \subset \mathbb{R}^d$ and again $|X| = (d + 1)(r - 1) = T(d, r) - 1$. We will show that if $d < \frac{rk}{r-k} - 1$ then for any r -partition of X ($X = C_1 \cup \dots \cup C_r$) there are k of the r sets whose convex hulls do not have a point in common.

W.l.o.g one may assume that none of the C_j -s intersects all the $d+1$ rays, i.e. that for every j , $I(C_j) \subsetneq D$. Indeed, suppose for instance that C_r intersects all the $d+1$ rays. Denote $X \setminus C_r$ by \tilde{X} . The set $\tilde{X} = \cup_{j=1}^{r-1} C_j$ contains $d+1$ rays, each ray now containing $r-2$ points at most. By assumption $d < \frac{rk}{r-k} - 1$ and since $\frac{rk}{r-k} - 1 < \frac{(r-1)k}{r-k-1} - 1$, \tilde{X} satisfies the conditions of the theorem. If the theorem holds for \tilde{X} then there are k sets among the C_j -s which their convex hulls do not intersect. Therefore, the theorem holds for X .

From now on we assume that for every j , $I(C_j) \subsetneq D$.

We now prove the theorem. To do this we define a (weight) function: given k of the sets (say C_{j_1}, \dots, C_{j_k}) and given a ray R_i we define:

$$W((C_{j_1}, \dots, C_{j_k}), R_i) := \begin{cases} 0 & \exists s \in \{1, \dots, k\} \text{ s.t. } R_i \cap C_{j_s} = \emptyset \\ 1 + \#\{s : |C_{j_s} \cap R_i| > 1\} & \text{otherwise} \end{cases}$$

In section 4.1 we establish the following result: if $\cap_{s=1}^k \text{conv}(C_{j_s}) \neq \emptyset$ then:

$$(4.0.4) \quad \sum_{i=0}^d W((C_{j_1}, \dots, C_{j_k}), R_i) \geq k$$

In section 4.2 we will show that for each $i \in \{0, \dots, d\}$:

$$(4.0.5) \quad \sum_{1 \leq j_1 < j_2 < \dots < j_k \leq r} W((C_{j_1}, \dots, C_{j_k}), R_i) \leq \binom{r-1}{k}$$

We use these two results to show theorem 1.0.5. If $\cap_{s=1}^k \text{conv}(C_{j_s}) \neq \emptyset$ for all $1 \leq j_1 < j_2 < \dots < j_k \leq r$ then from equation (4.0.4) and equation (4.0.5) we conclude:

$$k \binom{r}{k} \leq \sum_{1 \leq j_1 < j_2 < \dots < j_k \leq r} \sum_{i=0}^d W((C_{j_1}, \dots, C_{j_k}), R_i) \leq (d+1) \binom{r-1}{k}$$

We get the inequality,

$$k \binom{r}{k} \leq (d+1) \binom{r-1}{k}$$

which is equivalent to $d \geq \frac{rk}{r-k} - 1$ and the theorem follows.

4.1. A lower bound for the weight function W . Given k of the sets, say $\{C_j\}_{j \in J}$ ($J \subset \{1, \dots, r\}$, $|J| = k$), we would like to know if the intersection of their convex hulls is empty or not. Recall that these sets are all subsets of $\cup_{i=0}^d R_i$. We will be interested only in those rays which intersect all of the k sets, i.e. a ray R_i which will be of interest (a **mutual ray**) must satisfy $R_i \cap C_j \neq \emptyset$, $\forall j \in J$. This fact is expressed in proposition 4.1.1. For convenience we define $I(J) := \cap_{j \in J} I(C_j)$ which is the set of indices of the mutual rays. The number of mutual rays is $|I(J)|$. The union of all mutual rays is simply $R(I(J))$. Using this notation we can now state and prove:

Proposition 4.1.1. *For $J \subset \{1, \dots, r\}$ if $I(C_j) \subsetneq D$ for every $j \in J$ then*

$$\cap_{j \in J} \text{conv}(C_j) = \cap_{j \in J} \text{conv}(C_j \cap R(I(J)))$$

Proof. Notice that the union of all the rays of interest equals $R(I(J))$. The inclusion of the r.h.s in the l.h.s is obvious. We show that the l.h.s is included in the r.h.s as follows: suppose $x \in \cap_{j \in J} \text{conv}(C_j)$ then by proposition 2.1.2 $\text{supp}(x) \subset I(J)$. By proposition 2.1.3 it follows that $x \in \text{conv}(C_j \cap R(I(J)))$ for all $j \in J$. \square

Proposition 4.1.2. *Given k of the sets, $\{C_j\}_{j \in J}$ ($J \subset \{1, \dots, r\}$, $|J| = k$), if $|I(J)| = m$ and if each of the mutual rays R_i ($i \in I(J)$) contains exactly one point of each of the C_j -s then $m < k$ implies $\cap_{j \in J} \text{conv}(C_j) = \emptyset$*

Proof. Since $|I(J)| = m$ we have m mutual rays spanning an m dimensional linear space. Each of the sets C_j is composed of m linearly independent points and therefore spans an $m - 1$ dimensional affine space. Since the points are in general position, we have:

$$\dim \cap_{j \in J} \text{aff}(C_j) = m - \sum_{j \in J} \text{codim}(\text{aff}(C_j)) = m - k$$

\square

The following is a natural generalization of the last proposition:

Proposition 4.1.3. *Given k of the sets, $\{C_j\}_{j \in J}$ ($J \subset \{1, \dots, r\}$, $|J| = k$), suppose $|I(J)| = m$ and denote by t the number of sets among the C_j -s which contain more than one point of at least one of the mutual rays. In this case, $m < k - t$ implies $\cap_{j \in J} \text{conv}(C_j) = \emptyset$*

Proof. There are $k - t$ sets which satisfy the conditions of the previous proposition so the intersection of their convex hulls is empty. Therefore the intersection of all the k convex hulls is also empty. \square

From the last two claims, inequality 4.0.4 follows immediately.

4.2. An upper bound for the weight function W . Given a ray R_i , the weight of the ray is defined as:

$$W(R_i) := \sum_{1 \leq j_1 < j_2 < \dots < j_k \leq r} W((C_{j_1}, \dots, C_{j_k}), R_i)$$

We will show that $W(R_i)$ is maximal if each point in R_i belongs to a different set C_j , i.e. for every j , if $C_j \cap R_i \neq \emptyset$ then $|C_j \cap R_i| = 1$. If this condition holds, it is clear that $W(R_i) = \binom{r-1}{k}$ and inequality (4.0.5) follows. We now prove:

Proposition 4.2.1. *If for every j , $C_j \cap R_i \neq \emptyset$ implies $|C_j \cap R_i| = 1$ then $W(R_i)$ is maximal.*

Proof. Given any ray, e.g. R_0 , not all sets intersect R_0 since there are only $r - 1$ points on R_0 and r sets. Therefore, assume C_1 satisfies $C_1 \cap R_0 = \emptyset$. Suppose that there exist a set C_2 s.t. $|C_2 \cap R_0| > 1$. Choose $x \in C_2 \cap R_0$ and define a new partition of X which is identical to the old partition except that $(C_1)_{\text{new}} = C_1 \cup \{x\}$ and $(C_2)_{\text{new}} = C_2 \setminus \{x\}$. In other words, we move the element x from C_2 to C_1 . We claim that the weight of R_0 in the new partition is greater than its weight under the old partition.

There are two cases. If $|C_2 \cap R_0| > 2$, the weight clearly increases (notice that the expression $\#\{s : |C_{j_s} \cap R_0| > 1\}$ does not change if $|C_{j_s} \cap R_0|$ is greater than 2 or equal to 2). If $|C_2 \cap R_0| = 2$, denote by t the number of sets which intersect R_0 in the old partition, i.e. $t = \#\{j : C_j \cap R_0 \neq \emptyset\}$. On the one hand, by moving x

from C_2 to C_1 the weight of R_0 loses a point for every k -tuple which includes C_2 , i.e. the weight decreases by $\binom{t-1}{k-1}$. On the other hand, the weight of R_0 increases at least by one point for every k -tuple which includes C_1 , i.e. the weight increases by $\binom{t}{k-1}$. Therefore the total weight of R_0 increases at least by $\binom{t}{k-1} - \binom{t-1}{k-1} > 0$.

We have shown that to achieve optimal weight, R_0 should intersect each set at one point at most. \square

5. PROOF OF THEOREM 1.0.6: THE CASE $d = 5$, $r = 3$, $k = 2$

We start with the usual construction of X (as in theorem 1.0.3): six rays R_i , $i = 0, \dots, 5$ inside \mathbb{R}^5 with two points chosen on each ray. The set X contains $12 = T(5, 3) - 1$ points.

Consider partitions of X into three sets $X = C_1 \cup C_2 \cup C_3$. As in the case $d = 3, r = 4, k = 2$ we say that a partition is "good" if there are two sets C_i, C_j such that $\text{conv}(C_i) \cap \text{conv}(C_j) = \emptyset$ and is "bad" if for every $1 \leq i, j \leq 3$, $\text{conv}(C_i) \cap \text{conv}(C_j) \neq \emptyset$. Our aim in this section is to show that by moving three points of X we can turn all "bad" partitions into "good" partitions.

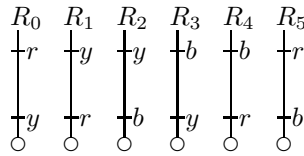
Proposition 5.0.2. *The "bad" partitions are exactly those partitions in which for every $1 \leq i < j \leq 3$ there exists exactly one ray where the lower point belongs to C_j and the higher point belongs to C_i and exactly one ray where the lower point belongs to C_i and the higher point belongs to C_j .*

Proof. Given a "bad" partition of $X = C_1 \cup C_2 \cup C_3$, recall that each ray contains exactly two points so that if ray R intersects C_i and C_j then $|R \cap C_i| = |R \cap C_j| = 1$.

By proposition 4.1.2 given two sets C_i, C_j there are at least two rays intersecting both C_i and C_j . Since there are three possible pairs of sets ($1 \leq i < j \leq 3$) and six rays there are exactly two rays intersection both C_i and C_j for every pair C_i, C_j .

From proposition 2.1.5 with $n = 2$ we get that in one of these rays C_i is lower than C_j and in the other, C_j is lower than C_i . \square

We see that the "bad" partitions are partitions in which for every $1 \leq i \leq 3$, $|C_i| = |I(C_i)| = 4$ and for every $1 \leq i < j \leq 3$, $|I(C_i) \cap I(C_j)| = 2$. Indeed, in these partitions $\text{conv}(C_i) \cap \text{conv}(C_j) \neq \emptyset$. Let us depict the following example of a "bad" partition:



Each ray is represented by a line starting at the origin o (the six different origins of the rays in the picture should of course identified as one) and we denote the different sets of the partition by colors: yellow, blue and red. The last proposition implies that *any* "bad" partition is the same as this partition up to a permutation of the rays. Therefore, we will dedicate some effort to study this partition.

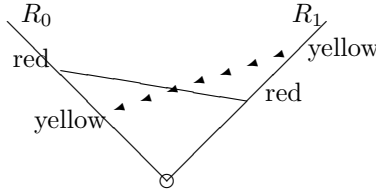
Before continuing with the proof, recall the non-negative representation of a point (equation 2.1.1): if p_0, \dots, p_d are the vertices of the simplex whose center is at the origin ($\sum_i p_i = 0$), each point $x \in \mathbb{R}^5$ can be uniquely represented by $x = \sum \xi_i p_i$

s.t. $\min(\xi_i) = 0$. A hyperplane in \mathbb{R}^5 will be represented by an equation

$$H = \{(\xi_0, \dots, \xi_5) \mid \sum_{i=0}^5 a_i \xi_i = \alpha\}$$

where we demand that $\sum_i a_i = 0$ so that the hyperplane H be well defined. Indeed, if $(\xi_0, \dots, \xi_5) = (\xi_0 + \lambda, \dots, \xi_5 + \lambda)$ are two different representation of a point $x \in \mathbb{R}^5$ then: $\sum_i a_i \xi_i = \sum_i a_i (\xi_i + \lambda)$ iff $\sum_i a_i = 0$.

Returning to our example of a "bad" partition, we notice that each of the color sets is a three dimensional simplex (inside \mathbb{R}^5) and that any two color sets intersect at a single point residing on an edge of each of them. For example the point of intersection of the red and yellow sets is in $\text{span}\{R_0, R_1\}$ as depicted below:



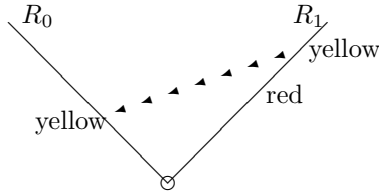
Moreover, any hyperplane of the form:

$$H = \{(\xi_0, \dots, \xi_5) \mid -a\xi_2 - b\xi_3 + c\xi_4 + d\xi_5 = 0\}$$

satisfying $a + b = c + d$, for $a, b, c, d > 0$, (weakly) separates the yellow set from the red set, where the red set is contained in $H \cup H_+$ and the yellow set is contained in $H \cup H_-$ ($\text{span}\{R_0, R_1\}$ is contained in H). By H_-, H_+ we refer to the two [[open/closed]] half spaces determined by H .

We start by moving the red point $P_{0,1}$ in order to separate the red set from the yellow set (remark: this will not change the intersection of the red and blue sets since the red and blue intersect in $\text{span}\{R_4, R_5\}$ and point $P_{0,1}$ is "far" from this plane). Define: $P'_{0,1} = P_{0,1} + \epsilon \vec{u}$ where:

- (1) ϵ is small enough so as to prevent the "good" partitions from becoming "bad". Such an ϵ exists as we saw in remark 3.0.3.
- (2) We choose \vec{u} such that there exists an H of the above form where $P'_{0,1} \in H_+$, i.e. there exist $a, b, c, d > 0$, $a + b = c + d$ such that $-au_2 - bu_3 + cu_4 + du_5 > 0$. This H contains R_0 and R_1 and the intersection of H with the red simplex and the yellow simplex are depicted below:



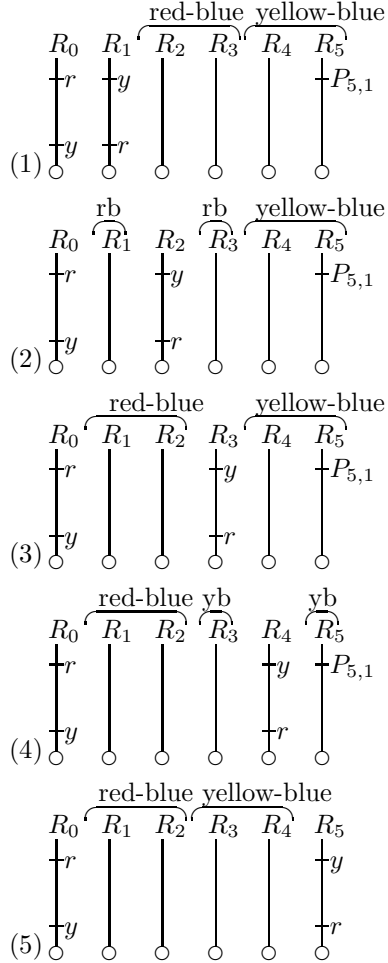
Finding such a \vec{u} indeed separates between the red and yellow sets: the red set is included in $H \cup H_+$ where H contains a single red point, the yellow set is included in $H \cup H_-$ where H contains a yellow segment and the red point and yellow segment do not intersect. The following lemma will aid us in finding \vec{u} :

Lemma 5.0.3. *given four numbers u_i, u_j, u_k, u_l if $\max\{u_i, u_j\} > \min\{u_k, u_l\}$ then there exist $0 < \alpha, \beta < 1$ such that $\alpha u_i + (1 - \alpha)u_j - \beta u_k - (1 - \beta)u_l > 0$.*

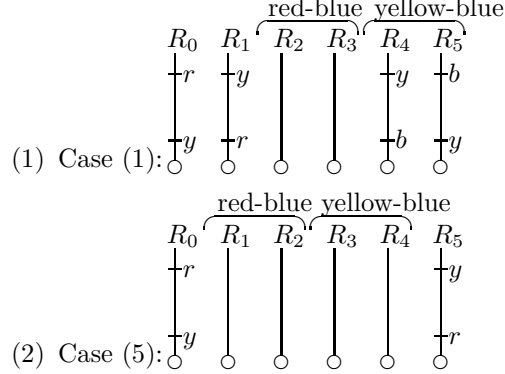
Proof. Since u_i, u_j, u_k, u_l are points in \mathbb{R}^1 , we can find $x_1 = \alpha u_i + (1 - \alpha)u_j \in \text{conv}\{u_i, u_j\}$ and $x_2 = \beta u_k + (1 - \beta)u_l \in \text{conv}\{u_k, u_l\}$ so that $x_1 > x_2$. \square

If k, l are the indices of the rays containing the yellow-blue points and i, j are the indices of the rays containing the red-blue points then moving $P_{0,1} \rightsquigarrow P'_{0,1} = P_{0,1} + \epsilon \vec{u}$ for any \vec{u} satisfying the demands of the lemma will separate the yellow set from the red set. Choose a specific $\vec{u} = (0, 1, 2, 3, 4, 5)$ and move $P_{0,1}$ to $P'_{0,1}$ along this \vec{u} . We separated the red simplex and yellow simplex in the specific coloring shown above. We claim, moreover, that this choice of \vec{u} separates many more partitions (in fact 100 of the possible 120 partitions).

Given another partition, w.l.o.g we can name the colors so that the ray R_0 is the same as in the previous example of a partition. Since $\vec{u} = (0, 1, 2, 3, 4, 5)$ then $u_i > u_j$ iff $i > j$ so we compare indices instead of coordinates. Hence, if the maximum of the indices of the red-blue rays is greater than the minimum of the indices of the yellow-blue rays then our choice of \vec{u} and the consequent move of $P_{0,1}$ separates the red simplex from the yellow simplex. Which of the partitions is still "bad": all those partitions for which the indices of the blue-yellow rays are greater than the indices of the red-blue rays. We specify all these cases:



We now turn to move point $P_{5,1}$: $P'_{5,1} = P_{5,1} + \epsilon'(5, 4, 3, 2, 1, 0)$. Our aim is to separate the blue simplex from the yellow simplex. Using the same arguments used in the separation of the red simplex from the yellow simplex we are left with the following cases:



We finish with a third move, aiming to separate the red simplex from the blue simplex. In all the remaining cases, R_2 contains blue and red points and R_4 contains yellow and blue. Therefore, the following move $P'_{2,1} = P_{2,1} + \epsilon''(1, 0, 0, 0, 1, 0)$ separates the red from the blue in the case where $P_{2,1}$ is red as well as in the case where $P_{2,1}$ is blue.

This concludes the proof of 1.0.6: we began with 12 points in \mathbb{R}^5 and perturbed three of them to get 12 points in \mathbb{R}^5 such that in any three partition of them there are two sets whose convex hull do not intersect.

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